REMARKS ON QUANTUM ERGODICITY

GABRIEL RIVIÈRE

Abstract. We prove a generalized version of the Quantum Ergodicity Theorem on smooth compact Riemannian manifolds without boundary. We apply it to prove some asymptotic properties on the distribution of typical eigenfunctions of the Laplacian in geometric situations where the Liouville measure is not (or not known to be) ergodic.

1. Introduction

Let M be a smooth, compact, connected Riemannian manifold of dimension d (without boundary). Denote by L the normalized Liouville measure on the unit cotangent bundle S^*M and by g^t the geodesic flow on S^*M. Let $(\psi_j)_{j\in\mathbb{N}}$ be an orthonormal basis of eigenfunctions of $-\Delta_g$ associated to a nondecreasing sequence of eigenvalues $(\lambda_j^2)_{j\in\mathbb{N}}$, i.e.

$$-\Delta_g\psi_j = \lambda_j^2\psi_j,\ \|\psi_j\|_{L^2(M)} = 1.$$

In the following, we will write $N(\lambda) := \sharp\{j : \lambda_j^2 \leq \lambda^2\}$. Our goal in this note is to describe the asymptotic distribution of this sequence of eigenfunctions as λ_j^2 tends to infinity. For that purpose, we introduce the following distribution on S^*M:

$$\forall a \in \mathcal{C}^\infty(S^*M),\ \mu_j(a) = \int_{S^*M} a d\mu_j := \langle \psi_j, \mathrm{Op}(a)\psi_j\rangle,$$

where $\mathrm{Op}(a)$ is a pseudodifferential operator with principal symbol a. It is a classical fact to check that any accumulation point[1] of this sequence belongs to the set $\mathcal{M}(S^*M, g^t)$ of $(g^t)_t$-invariant probability measures on S^*M [3].

If the Liouville measure is ergodic, the Quantum Ergodicity Theorem states that there exists a subset S of density[2] 1 in $\mathbb{N}$ such that the sequence $(\mu_j)_{j\in S}$ converges to the Liouville measure L [14, 18, 4]. In other words, it means that the eigenfunctions become equidistributed in S^*M. We refer the reader to [20] for a recent detailled survey on related issues. In our context, the main example of application is given by geodesic flows on manifolds of negative curvature or more generally by uniformly hyperbolic geodesic flows: in this setting, the Liouville measure is known to be ergodic.

Here, we are interested in the case where we drop the ergodicity assumption. The main examples we have in mind are geodesic flows for which the Liouville measure is ergodic on a subset of positive measure. For instance, this kind of situations occurs when the geodesic flow is supposed to be nonuniformly hyperbolic for the Liouville measure [1]. In these cases, we derive properties on the asymptotic distributions of the eigenmodes. For that purpose, we prove an alternative version of the Quantum Ergodicity Theorem that does not rely on ergodicity. Then, we apply this result in several geometric contexts. For example, we obtain a kind of equidistribution property for subsequences of eigenfunctions of the Laplacian on surfaces of nonpositive curvature with genus ≥ 2. In this setting, the Liouville measure is not known to be ergodic; thus, the standard Quantum Ergodicity Theorem does not apply a priori.

Date: October 12, 2012.

[1] Convergence is for the standard topology on $\mathcal{D}'(S^*M)$.

[2] Recall that $S \subset \mathbb{N}$ have density 1 if $\lim_{n\to+\infty} \frac{1}{n}\sharp\{k \in S : 1 \leq k \leq n\} = 1$.

2. Statement of the main result

Thanks to the Birkhoff Ergodic Theorem [6], there exists a subset $\Lambda \subset S^*M$ such that $L(\Lambda) = 1$ and, for every ρ in Λ,

$$\lim_{T\rightarrow+\infty}\frac{1}{T}\int_0^T \delta_{g^t\rho}dt = L_{\rho},$$

where L_{ρ} is a $(g^t)_t$-invariant probability measure on S^*M and where the convergence is for the weak-$\star$ topology, i.e.

$$\forall a \in \mathcal{C}^0(S^*M),\ \forall \rho\in\Lambda,\ \lim_{T\rightarrow+\infty}\frac{1}{T}\int_0^T a(g^t\rho)dt = L_{\rho}(a).$$

Thanks to this property, we introduce $\text{Cv}(L)$ which is the closure (in $\mathcal{D}'(S^*M)$) of the convex hull of $\{L_{\rho} : \rho\in\Lambda\}$. We emphasize that the set $\text{Cv}(L)$ depends implicitely on the choice of Λ and that any element in $\text{Cv}(L)$ is a probability measure on S^*M invariant under the geodesic flow. One can then show the following version of the Quantum Ergodicity Theorem:

Theorem 2.1. *Let $(\psi_j)_{j\in\mathbb{N}}$ be an orthonormal basis of eigenfunctions of $-\Delta_g$ associated to a nondecreasing sequence of eigenvalues $(\lambda_j^2)_{j\in\mathbb{N}}$, i.e.*

$$-\Delta_g\psi_j = \lambda_j^2\psi_j,\ \|\psi_j\|_{L^2(M)} = 1.$$

Then, there exists $S\subset\mathbb{N}$ of density 1 such that any accumulation point of the sequence $(\mu_j)_{j\in S}$ belongs to $\text{Cv}(L)$.

We underline that this theorem is true for any orthonormal basis of eigenfunctions of Δ_g and that we do not make any particular assumption on the manifold (like ergodicity for instance). Even if this generalization is quite natural, we did not find any trace of such a result in the literature. If M is the sphere $\mathbb{S}^d$ endowed with its canonical metric, the set $\text{Cv}(L)$ is equal to $\mathcal{M}(S^*M, g^t)$ and the result is empty as we already know that any accumulation point of the sequence $(\mu_j)_{j\geq 0}$ is an invariant probability measure. In the "opposite" case where the Liouville measure is ergodic for the geodesic flow, we recover the standard Quantum Ergodicity Theorem [14, 18, 4] as $\text{Cv}(L)$ can be chosen equal to $\{L\}$ (for a good subset Λ).

In the physics literature, the "semiclassical eigenfunctions hypothesis" states that the eigenmodes $(\psi_j)_{j\geq 0}$ must be asymptotically concentrated into regions of phase space which a typical orbit explores in the long time limit [11, 2]. In our context, the set Λ could represent in some sense a set of typical orbits and the measure L_{ρ} is the canonical measure associated to the orbit of a point ρ in the phase space S^*M.

Regarding this conjecture, it seems natural to understand when there exist a subset Λ of full measure and a typical subsequence of eigenmodes $(\psi_j)_{j\in S}$ such that the accumulation points of $(\mu_j)_{j\in S}$ are exactly given by $\{L_{\rho} : \rho\in\Lambda\}$: this question was for instance raised by Shnirelman in [15] – end of paragraph $AD.2$. In such generality, it is a priori not true as there exist geometric situations where the set $\{L_{\rho} : \rho\in\Lambda\}$ cannot not be reduced to L while there exists a typical family of states that converges to L [19, 20, 8, 21].

Our theorem shows that, for a typical family $(\mu_j)_{j\in S}$, the accumulation points belong to a larger set than $\{L_{\rho} : \rho\in\Lambda\}$, precisely they belong to the closure of its convex hull. We underline that results related to these questions were also obtained by Marklof and O'Keefe for specific families of quantum maps with divided phase space [9].

We will explain in paragraph 4 how one can get our generalized version of the Quantum Ergodicity Theorem by implementing an idea used by Sjöstrand in the context of damped wave equations [16]. In fact, our proof will combine Hahn-Banach theorem with the following main lemma that makes the connection with the results in [16] more explicit:

Lemma 2.2. *Let a be an element in $\mathcal{C}^{\infty}(S^*M,\mathbb{R})$. Then, there exists $S\subset\mathbb{N}$ of density 1 such that,*

$$\text{essinf}\ \ L_{\rho}(a) \leq \liminf_{j\rightarrow+\infty, j\in S}\mu_j(a) \leq \limsup_{j\rightarrow+\infty, j\in S}\mu_j(a) \leq \text{esssup}\ \ L_{\rho}(a).$$

Organization of the following. In the next paragraph, we apply our theorem in several geometric situations. After that, we give the proof of theorem 2.1 and provide the proof of an intermediary proposition that we used in our applications.

In the following, we will sometimes use the notation $\mathcal{D}(S^*M)$ for the space $\mathcal{C}^\infty(S^*M)$ when we want to emphasize that we are working with distributions.

3. Application of theorem 2.1

Before entering the details of the proof, we describe several geometric situations where the Liouville measure is a priori not supposed to be ergodic and we apply theorem 2.1 in order to derive some weak equidistribution properties for the eigenfunctions.

3.1. **Nonuniformly hyperbolic geodesic flows.** A direct consequence of theorem 2.1 is the following property:

Corollary 3.1. *Suppose there exists I at most countable and a family $(\Lambda_i)_{i\in I}$ of invariant subsets such that*

- $L(\Lambda_i) > 0$ *for every* $i \in I$*;*
- $\Lambda_i \cap \Lambda_j$ *is empty when* $i \neq j$*;*
- $L(\cup_i \Lambda_i) = 1$*;*
- $L_{|\Lambda_i}$ *is ergodic for every* $i \in I$*.*

Then, there exists a subset $S \subset \mathbb{N}$ of density 1 such that any accumulation point of the sequence $(\mu_j)_{j\in S}$ is absolutely continuous with respect to L.

Theorem 2.1 applies in this setting as in this case one can take $\Lambda = \cup_{i\in I}\Lambda_i$ (up to some subset of measure 0) and verify that

$$\mathrm{Cv}(L) := \left\{ \sum_{i\in I} t_i \frac{L_{|\Lambda_i}}{L(\Lambda_i)} : \ \forall i \in I, \ 0 \leq t_i \leq 1 \text{ and } \sum_{i\in I} t_i = 1 \right\}.$$

Thus, if the phase space can be divided into (at most) countably many subsets on which L is ergodic, then most of the eigenmodes are equidistributed, in the sense that they have to converge to a convex combination of $(L_{|\Lambda_i}/L(\Lambda_i))_{i\in I}$. We emphasize that no hypothesis on the nonuniform hyperbolicity is made in the above statement. However, we recall that, thanks to Pesin works, the assumptions of the corollary are automatically satisfied when *the geodesic flow is nonuniformly hyperbolic with respect to the Liouville measure* – Theorem 11.5 in [1].

In [8], Gutkin constructed a nonuniformly hyperbolic geodesic flow on a billiard table for which the Liouville measure is not ergodic. However, his system presented a symmetry that allowed him to obtain a stronger result than corollary 3.1, precisely he proved the existence of a subsequence of density 1 converging to the Liouville measure. It is not clear to the author whether there exist smooth compact Riemannian manifolds without boundary which satisfy the assumption of the corollary with $|I| \geq 2$ and which do not present a symmetry like the one in [8]. For instance, we do not know if ergodicity is a "generic property" in the family of nonuniformly hyperbolic geodesic flows.

3.2. **Geodesic flows with divided phase space.** In [5] – section 11, Donnay constructs Riemmanian metrics on the sphere $\mathbb{S}^2$ for which the phase space splits into a chaotic component and an integrable one. His idea is to remove three or more points from the sphere and to endowe the induced punctured surface with the Poincaré metric; then, he attaches smoothly a so called "light-bulb cap" in a neighborhood of each deleted point. We will call these spheres Donnay's surfaces.

The geodesic flow he obtains is not ergodic for L. Yet, the phase space S^*M contains two disjoint invariant subsets Λ_{chaotic} and $\Lambda_{\text{integrable}}$ of *positive Liouville measure* satisfying $L(\Lambda_{\text{chaotic}} \cup \Lambda_{\text{integrable}}) = 1$. More precisely, $\Lambda_{\text{integrable}}$ consists of orbits that stay in the caps and $g^t_{|\Lambda_{\text{chaotic}}}$ is nonuniformly hyperbolic for the measure $L_{|\Lambda_{\text{chaotic}}}$. Thanks to Pesin Theorem, this chaotic part of the phase space can be divided into (at most) countably many subsets $(\Lambda_i)_{i\in I}$ of positive Liouville measure such that $L_{|\Lambda_i}$ is ergodic for every i in I.

If we apply theorem 2.1 in this geometric context, there exists a subset $S \subset \mathbb{N}$ of density 1 such that the accumulations points of the sequence $(\mu_j)_{j\in S}$ are of the form

$$\mu = \alpha \sum_{i\in I} t_i \frac{L_{|\Lambda_i}}{L(\Lambda_i)} + (1-\alpha)\nu_{\text{int}}, \tag{1}$$

where $0 \leq \alpha \leq 1$, $0 \leq t_i \leq 1$, $\sum_{i\in I} t_i = 1$ and ν_{int} belongs to the closure of the convex hull of $\{L_\rho : \rho \in \Lambda_{\text{integrable}}\}$.

3.3. **More on divided phase space.** At this point, our different statements do not forbid that the ergodic component associated to a subset Λ_i of positive Liouville measure has a weight $t_i = 0$. This motivates the following proposition which is valid for any smooth and compact Riemannian manifold without boundary and which gives partial informations on this question.

Proposition 3.2. *Let M be a smooth, compact, connected Riemannian manifold without boundary. Suppose there exists an invariant subset Λ' such that $L(\Lambda') > 0$ and such that $L_{|\Lambda'}$ is ergodic. Then, the following properties hold:*

1. *There exists a subset S of density 1 in $\mathbb{N}$ such that any accumulation point of the sequence $(\mu_j)_{j\in S}$ is of the form*
$$\alpha \frac{L_{|\Lambda'}}{L(\Lambda')} + (1-\alpha)\nu_0,$$
where $0 \leq \alpha \leq 1$ and ν_0 belongs to the closure of the convex hull of $\{L_\rho : \rho \in \Lambda_0\}$ for some $\Lambda_0 \subset \Lambda'^c$ satisfying $L(\Lambda'^c) = L(\Lambda_0)$.
2. *Assume that Λ' contains a nonempty open ball (modulo 0).*
Then, for every $0 \leq \delta < L(\Lambda')$, there exists a subset $S_\delta \subset S$ of density $\geq \delta$ such that, for any accumulation point of the sequence $(\mu_j)_{j\in S_\delta}$, one has
$$0 < \frac{L(\Lambda') - \delta}{1-\delta} \leq \alpha \leq 1.$$

This result is true for any orthonormal basis of eigenfunctions of Δ_g. The first part is a direct application of theorem 2.1 while the second part can be obtained as an application of lemma 2.2 – see paragraph 5 for details. This proposition gives us a sufficient condition to observe a kind of equidistribution property on an invariant subset Λ' – see [15] for related questions.

Remark 3.3. We emphasize that if τ_0 is an invariant probability measure such that $L(\{\rho : L_\rho = \tau_0\}) > 0$, then τ_0 must be of the form $\frac{L_{|\Lambda'}}{L(\Lambda')}$ for some invariant subset Λ'.

Remark 3.4. Galkowski recently proved similar equidistribution properties for some class of systems with divided phase space [7]. Precisely, he considered the case of manifolds with piecewise smooth boundaries and he obtained an analogue of part (1) of this proposition with Λ' satisfying $L_{|\Lambda'}$ ergodic, $L(\Lambda') > 0$ and $L(\partial\Lambda' - \Lambda') = 0$. It would be interesting to understand if theorem 2.1 could also be proved in the setting of manifolds with piecewise smooth boundaries.

Remark 3.5. Under this form, the proposition cannot be directly applied to treat the case of Donnay's surfaces as $|I|$ could be a priori ≥ 2 in equation (1). At the end of paragraph 5, we will check that the statement can be generalized to fit to this example. Precisely, in the setting of Donnay's surfaces, we will obtain the existence of a subset S' of positive density such that any accumulation point of the sequence $(\mu_j)_{j\in S'}$ is of the form given by equation (1) with $\alpha > 0$. In other words, it means that a positive proportion of eigenmodes are equidistributed in the chaotic part of the phase space for Donnay's surfaces.

3.4. **Surfaces of nonpositive curvature.** We will now give an application of the previous proposition in the context of nonpositively curved manifolds. We suppose that M is a *surface of nonpositive curvature* of genus ≥ 2. For any point x in M, we will denote by $K(x) \leq 0$ the sectional curvature at point x. We will make a small abuse of notations and use also the notation K for its

canonical lift on S^*M. Following [1], we introduce the following subset

$$\Lambda' := \left\{ \rho \in S^*M : \limsup_{T \to +\infty} \frac{1}{T} \int_0^T K \circ g^t(\rho) dt < 0 \right\}. \tag{2}$$

Thanks to [12, 1], the set Λ' is *open (modulo* 0*) and everywhere dense* and it satisfies $L(\Lambda') > 0$ and $L_{|\Lambda'}$ is ergodic. We underline that it is still an open question to determine whether $L(\Lambda') = 1$ or not for any surface of nonpositive curvature of genus ≥ 2. In other words, *it is not known if the Liouville measure is ergodic or not.* However, we can prove that the eigenfunctions satisfy some equidistribution properties in this negatively curved part of the surface. In fact, the subset Λ' satisfies all the requirements of proposition 3.2 and precisely, we have

Corollary 3.6. *If M is a surface of nonpositive curvature $K(x)$ and of genus ≥ 2, then the conclusions of proposition 3.2 are satisfied with Λ' defined by* (2).

This result is true for any orthonormal basis of eigenfunctions of Δ_g. This corollary tells us that a positive proportion of eigenmodes are asymptotically equidistributed in the set Λ' even if we do not have ergodicity of the Liouville measure on the entire phase space.

Remark 3.7. We emphasize that our result does not forbid that the eigenmodes put also some weight in the region $\{K = 0\}$. In fact, one can apply the Birkhoff Ergodic Theorem and find that almost everywhere on Λ'^c, $L_\rho(K) = 0$ which implies that the measure ν_0 in the conclusion of proposition 3.2 satisfies $\nu_0(K) = 0$.

If we project the distributions μ_j on the base, we find the following notable consequence of this corollary: any accumulation point of the sequence $(K|\psi_j|^2 \mathrm{vol}_M)_{j \in S}$ is of the form $cK\mathrm{vol}_M$ where $c \geq 0$ is a constant. Moreover, there are subsequences for which c can be chosen positive.

Remark 3.8. In the case where $\dim M \geq 2$, one can introduce the following subset of S^*M:

$$\Lambda' := \left\{ \rho \in S^*M : \limsup_{T \to +\infty} \frac{1}{T} \int_0^T K_{\pi \circ g^t \rho}(g^t \rho, g^t \rho') dt < 0, \text{ for every } \rho' \text{ orthogonal to } \rho \right\},$$

where $\pi : S^*M \to M$ is the canonical projection on M and $K_x(v_1, v_2)$ is the sectional curvature for x in M and v_1, v_2 in T_x^*M. Suppose now that M has nonpositive curvature, i.e. $K_x(v_1, v_2) \leq 0$ for every x in M and every v_1, v_2 in T_x^*M. Under some extra geometric assumptions[3] on M that are always satisfied by nonpositively curved surfaces of genus ≥ 2, the set Λ' is again open (modulo 0) and everywhere dense and it satisfies $L(\Lambda') > 0$ and $L_{|\Lambda'}$ is ergodic. Then, corollary 3.6 can be extended in $\dim M \geq 2$ modulo the above extra geometric assumptions.

3.5. **Flat torus.** In this paragraph, we apply theorem 2.1 to the flat torus $\mathbb{T}^d$ for which there is no subset Λ' of positive measure on which L is ergodic. We introduce the subset of "irrational" vectors

$$\Lambda := \left\{ (x, \xi) \in S^*\mathbb{T}^d : \ \forall p \in \mathbb{Z}^d - \{0\}, \ p.\xi \neq 0 \right\}.$$

This set has full Liouville measure and for every $\rho = (x, \xi)$ in Λ, $L_\rho = dx \times \delta_\xi$. In particular, the projection on $\mathbb{T}^d$ of any element in $\mathrm{Cv}(L)$ is the Lebesgue measure dx. Applying theorem 2.1, we obtain the following corollary:

Corollary 3.9. *For any orthonormal basis $(\psi_j)_{j \in \mathbb{N}}$ of eigenfunctions of Δ on $\mathbb{T}^d$, there exists a subset S of density* 1 *in $\mathbb{N}$ such that*

$$\forall a \in \mathcal{C}^0(\mathbb{T}^d), \quad \lim_{j \to +\infty, j \in S} \int_{\mathbb{T}^d} a(x) |\psi_j(x)|^2 dx = \int_{\mathbb{T}^d} a(x) dx.$$

This result is the analogue on $\mathbb{T}^d$ of Marklof-Rudnick's recent result on equidistribution of eigenfunctions on rational polygons [10].

[3]For more details on these assumptions, we refer the reader to [1], sections 2 and 17.

4. Proof of the main result

The proof follows classical ideas taken from [14, 18, 4, 16] that we carefully combine to get our generalized version of the Quantum Ergodicity Theorem. Without loss of generality, one can suppose that the sequence of distributions μ_j is real valued, i.e. $\mu_j(a)$ belongs to $\mathbb{R}$ when a is real valued.

4.1. **Proof of lemma 2.2.** We start our proof by giving the proof of the main lemma 2.2. Let a be an element in $\mathcal{C}^{\infty}(S^*M,\mathbb{R})$. We introduce the average of a at time T,

$$a_T(\rho):=\frac{1}{T}\int_0^T a\circ g^t(\rho)dt.$$

In order to simplify the presentation, denote $A_0:=\operatorname{esssup}_{\rho}L_{\rho}(a)$. By definition, one has that, for every $\delta>0$,

$$L\left(\{\rho\in S^*M : a_T(\rho)\geq A_0+\delta\}\right)\rightarrow 0,\ \text{as } T\rightarrow+\infty.$$

We start our proof by replacing Op by a positive quantization Op^+ that satisfies

$$b\geq 0\Longrightarrow \mathrm{Op}^+(b)\geq 0.$$

For instance, as in [4], one can take the so-called Friedrichs quantization. We have then, as j tends to $+\infty$

$$\mu_j(a)=\langle\psi_j,\mathrm{Op}^+(a)\psi_j\rangle+o(1).$$

Fix now $T>0$ and $\epsilon>0$. The Egorov theorem tells us that, as j tends to ∞,

$$\mu_j(a)=\langle\psi_j,\mathrm{Op}^+(a)\psi_j\rangle+o(1)=\langle\psi_j,\mathrm{Op}^+(a_T)\psi_j\rangle+o_T(1),$$

where the remainder depends on T. As in [16], one can define a new smooth function $\tilde{a}_T\leq a_T$ on S^*M such that

- $\tilde{a}_T(\rho)=a_T(\rho)$ when $a_T(\rho)\leq A_0+\frac{\sqrt{\epsilon}}{2}$.
- $\tilde{a}_T(\rho)\leq A_0+\sqrt{\epsilon}$ otherwise.

Remark 4.1. This function can be chosen in such a way that $\|\tilde{a}_T-a_T\|_{\infty}$ is bounded independently of $T>0$.

As j tends to infinity, one has the following equality:

$$\mu_j(a)=\langle\psi_j,\mathrm{Op}^+(a_T-\tilde{a}_T)\psi_j\rangle+\langle\psi_j,\mathrm{Op}^+(\tilde{a}_T)\psi_j\rangle+o_T(1).$$

The idea of introducing this new function is taken from [16] where it was used to study spectral asymptotics of the damped wave equation. In the following lines, we will show that

- most of the terms in the sequence $(\langle\psi_j,\mathrm{Op}^+(a_T-\tilde{a}_T)\psi_j\rangle)_{j\geq 0}$ are small following arguments from [18, 4];
- the other term in the RHS will be less than $A_0+\sqrt{\epsilon}+o_T(1)$ by construction.

A careful combination of these two facts will finally allow us to get our conclusion.

From trace asymptotics – see [4], paragraph 4 for instance, one has

$$\lim_{\lambda\rightarrow+\infty}\frac{1}{N(\lambda)}\sum_{\lambda_j^2\leq\lambda^2}\langle\psi_j,\mathrm{Op}^+(a_T-\tilde{a}_T)\psi_j\rangle=\int_{S^*M}(a_T-\tilde{a}_T)dL,$$

where each term in the sum is nonnegative (as $a_T-\tilde{a}_T\geq 0$). Thanks to our construction, one has

$$\int_{S^*M}(a_T-\tilde{a}_T)dL\leq C_{a,\epsilon}L\left(\left\{\rho\in S^*M : a_T(\rho)\geq A_0+\frac{\sqrt{\epsilon}}{2}\right\}\right).$$

Fix $\eta>0$. There exists $T_{\epsilon,\eta}>0$ such that, for every $T\geq T_{\epsilon,\eta}$, one can find $\lambda_T>0$ satisfying

$$\lambda\geq\lambda_T\Longrightarrow\frac{1}{N(\lambda)}\sum_{\lambda_j^2\leq\lambda^2}\langle\psi_j,\mathrm{Op}^+(a_T-\tilde{a}_T)\psi_j\rangle\leq\eta\epsilon.$$

We now fix $T = T_{\epsilon,\eta}$. Denote $D_\epsilon := \{j : \lambda_j^2 \leq \lambda^2 \text{ and } \langle \psi_j, \mathrm{Op}^+(a_T - \tilde{a}_T)\psi_j\rangle \geq \sqrt{\epsilon}\}$. Thanks to the Tchebychev inequality, we obtain that $\frac{\sharp D_\epsilon}{N(\lambda)} \leq \eta\sqrt{\epsilon}$ for $\lambda \geq \lambda_T$. This means that most of the terms in the sequence of nonnegative numbers $(\langle \psi_j, \mathrm{Op}^+(a_T - \tilde{a}_T)\psi_j\rangle)_{j\geq 0}$ are small.

As Op^+ is positive and $\tilde{a}_T \leq A_0 + \sqrt{\epsilon}$, one has that, for λ_j^2 larger than some $A > 0$, the term $\langle \psi_j, \mathrm{Op}^+(\tilde{a}_T)\psi_j\rangle + o_T(1)$ is less than $A_0 + 2\sqrt{\epsilon}$. Thanks to the above discusion, we can write

$$\sharp\left\{j : \lambda_j^2 \leq \lambda^2 \text{ and } \mu_j(a) \leq A_0 + 3\sqrt{\epsilon}\right\} \geq \sharp\left\{j : \lambda_j^2 \leq A \text{ and } \mu_j(a) \leq A_0 + 3\sqrt{\epsilon}\right\}$$
$$+\sharp\left\{j : A \leq \lambda_j^2 \leq \lambda^2 \text{ and } \langle \psi_j, \mathrm{Op}^+(a_T - \tilde{a}_T)\psi_j\rangle < \sqrt{\epsilon}\right\}.$$

If we denote $S_\epsilon := \left\{j : \lambda_j^2 \leq \lambda^2 \text{ and } \mu_j(a) \leq A_0 + 3\sqrt{\epsilon}\right\}$, then we have $\lim_{\lambda\to+\infty} \frac{\sharp S_\epsilon}{N(\lambda)} \geq 1 - \eta\sqrt{\epsilon}$. This is true for any $\eta > 0$ which implies that S_ϵ has density 1.

Using the procedure of paragraph 5 in [4], one can then obtain[4] a subset $S_0 \subset \mathbb{N}$ of density 1, such that any accumulation point of the sequence $(\mu_j(a))_{j\in S_0}$ is $\leq A_0$. This achieves the proof of the upper bound in lemma 2.2 and the lower bound can be easily derived by considering $-a$.

4.2. **Proof of theorem 2.1.** We are now in position to prove theorem 2.1. For that purpose, we interpret lemma 2.2 as an inequality on linear forms and then we apply Hahn-Banach Theorem.

First, we observe that

$$\inf_{\rho\in\Lambda} L_\rho(a) \leq \text{essinf } L_\rho(a) \leq \text{esssup } L_\rho(a) \leq \sup_{\rho\in\Lambda} L_\rho(a).$$

Fix now $(a_k)_{k\in\mathbb{N}}$ a family of smooth functions which is dense in $\mathcal{C}^0(S^*M,\mathbb{R})$ (for the uniform topology). Combining lemma 2.2 to the procedure of [4] (paragraph 5), one can choose a subset S of density 1 such that

$$\forall k \in \mathbb{N},\ \inf_{\rho\in\Lambda} L_\rho(a_k) \leq \liminf_{j\to+\infty, j\in S} \mu_j(a_k) \leq \limsup_{j\to+\infty, j\in S} \mu_j(a_k) \leq \sup_{\rho\in\Lambda} L_\rho(a_k).$$

Fix now an accumulation point μ of the sequence $(\mu_j)_{j\in S}$. By a density argument, the above inequality implies then

$$\forall a \in \mathcal{D}(S^*M,\mathbb{R}),\ \inf_{\rho\in\Lambda} L_\rho(a) \leq \mu(a) \leq \sup_{\rho\in\Lambda} L_\rho(a).$$

As the space $\mathcal{D}(S^*M,\mathbb{R})$ is the topological dual of $\mathcal{D}'(S^*M,\mathbb{R})$ (Theorem XIV, Chapter 3 in [13]), the previous inequality implies that, for every continuous linear form Φ on $\mathcal{D}'(S^*M,\mathbb{R})$,

$$\inf_{\rho\in\Lambda} \Phi(L_\rho) \leq \Phi(\mu) \leq \sup_{\rho\in\Lambda} \Phi(L_\rho).$$

Suppose by contradiction that μ does not belong to $\mathrm{Cv}(L)$. By the Hahn-Banach theorem [17], there exists a continuous linear form Φ_0 on $\mathcal{D}'(S^*M,\mathbb{R})$ that strictly separates the compact convex subset $\mathrm{Cv}(L)$ from $\{\mu\}$. In particular, there exists α in $\mathbb{R}$ such that

$$\forall \nu \in \mathrm{Cv}(L),\ \Phi_0(\nu) < \alpha \leq \Phi_0(\mu).$$

As $\mathrm{Cv}(L)$ is a compact subset of $\mathcal{D}'(S^*M,\mathbb{R})$, we get that $\sup_{\nu\in\mathrm{Cv}(L)} \Phi_0(\nu) < \alpha \leq \Phi_0(\mu)$. In particular, $\sup_{\rho\in\Lambda} \Phi_0(L_\rho) < \alpha \leq \Phi_0(\mu)$ which leads to the contradiction.

5. Proof of proposition 3.2

In this final section, we will prove proposition 3.2 that we used in our applications to surfaces of nonpositive curvature and under a generalized form to Donnay's surfaces – see paragraph 5.3 below.

As in the statement of the proposition, we fix an invariant subset Λ' in S^*M such that $L(\Lambda') > 0$ and $L_{|\Lambda'}$ is ergodic.

Thanks to theorem 2.1, there exists $S \subset \mathbb{N}$ of density 1 such that any accumulation point of the sequence $(\mu_j)_{j\in S}$ is of the form

$$\alpha\frac{L_{|\Lambda'}}{L(\Lambda')} + (1-\alpha)\nu_0,$$

[4]The set S_0 is constructed from the family of subsets $(S_{\frac{1}{l}})_{l\geq 1}$ we have just defined.

where $\alpha\geq 0$ and where ν_0 belongs to the closure of the convex hull of $\{L_{\rho}:\rho\in\Lambda_0\}$ for some $\Lambda_0\subset\Lambda'^c$ satisfying $L(\Lambda'^c)=L(\Lambda_0)$.

Suppose now that Λ' contains a nonempty open ball (modulo a set of zero Liouville measure). One can pick $\chi\geq 0$ a smooth function which is compactly supported in this open ball and which is not equal to 0 everywhere. The function χ satisfies then $L_{\rho}(\chi)=0$ almost everywhere on Λ'^c and $\int_{S^*M}\chi dL>0$. We underline that the properties of ν_0 and χ imply $\nu_0(\chi)=0$. Our goal is to show that there exist subsequences for which α can be chosen positive. For that purpose, we will study the limit of the sequence $(\mu_j(\chi))_{j\in S}$ and show that it must be positive for some subsequences of positive density.

5.1. **Preliminary remark.** Recall from Birkhoff Ergodic Theorem that

$$L_{\rho}(\chi)=\lim_{T\rightarrow+\infty}\frac{1}{T}\int_0^T\chi\circ g^t(\rho)dt$$

is well defined for L almost every ρ in S^*M. From our assumptions on Λ', one can verify that

$$L_{\rho}(\chi)=\frac{1}{L(\Lambda')}\int_{\Lambda'}\chi dL,\ \text{a.e. on }\Lambda'.$$

Moreover, recall that $L_{\rho}(\chi)=0$ almost everywhere on Λ'^c. Still thanks to Birkhoff Ergodic Theorem, we also have

$$\int_{S^*M}\chi dL=\int_{S^*M}L_{\rho}(\chi)dL(\rho)=\int_{\Lambda'}\chi dL.$$

5.2. **Proof of the proposition.** First, we observe that one can again replace Op by a nonnegative quantization procedure Op^+. In particular, one gets, as $j\rightarrow+\infty$,

$$\mu_j(\chi)=\langle\psi_j,\mathrm{Op}^+(\chi)\psi_j\rangle+o(1),$$

and $\langle\psi_j,\mathrm{Op}^+(\chi)\psi_j\rangle\geq 0$ (as $\chi\geq 0$). Thus, without loss of generality, one can look at the accumulation points of the sequence

$$\mu_j^+(\chi):=\langle\psi_j,\mathrm{Op}^+(\chi)\psi_j\rangle,\ j\in S,$$

where $S\subset\mathbb{N}$ is of density 1. Thanks to the trace asymptotics of paragraph 4 in [4], we observe that

$$\lim_{\lambda\rightarrow+\infty}\frac{1}{N(\lambda)}\sum_{j\in S:\lambda_j^2\leq\lambda^2}\mu_j^+(\chi)=\lim_{\lambda\rightarrow+\infty}\frac{1}{N(\lambda)}\sum_{j:\lambda_j^2\leq\lambda^2}\mu_j^+(\chi)=\int_{S^*M}\chi dL=\int_{\Lambda'}\chi dL.$$

Moreover, using lemma 2.2 and our remarks on the value of $L_{\rho}(\chi)$, one can find a subset $S'\subset S$ of density 1 such that any accumulation point of the sequence $(\mu_j^+(\chi))_{j\in S'}$ belongs to the interval

$$[\operatorname{essinf}L_{\rho}(\chi),\operatorname{esssup}L_{\rho}(\chi)]=\left[0,\frac{\int_{\Lambda'}\chi dL}{L(\Lambda')}\right].$$

We introduce the notation

$$\alpha_j:=\frac{\mu_j^+(\chi)}{\int_{\Lambda'}\chi dL}\geq 0.$$

In order to prove our proposition, it remains to verify that, for every $0<\epsilon\leq 1$, there exists a subset $S_{\epsilon}\subset S'$ of density $\geq\frac{1-\epsilon}{L(\Lambda')^{-1}-\epsilon}$ such that any accumulation point of the subsequence $(\alpha_j)_{j\in S_{\epsilon}}$ belongs to the interval $[\epsilon,L(\Lambda')^{-1}]$. The proof is quite straightforward: we briefly explain it for the sake of completeness.

Fix now $0<\epsilon\leq 1$. Let $\eta\ll\epsilon$ be a small positive number. From the properties of the sequence $(\alpha_j)_{j\in S'}$, there exists $A>0$ (depending on η) such that, for $\lambda^2\geq A$,

$$1-\eta\leq\frac{1}{N(\lambda)}\sum_{j\in S':\lambda_j^2\leq\lambda^2}\alpha_j\text{ and }(j\in S'\text{ and }\lambda_j^2>A\Longrightarrow\alpha_j\leq L(\Lambda')^{-1}+\eta).$$

Thus, one gets, for $\lambda^2 \geq A$,

$$1-\eta \leq \frac{1}{N(\lambda)} \sum_{j\in S':\lambda_j^2\leq A} \alpha_j + \epsilon \frac{1}{N(\lambda)} \sharp\left\{j\in S' : A<\lambda_j^2\leq \lambda^2 \text{ and } \alpha_j<\epsilon\right\}$$

$$+(L(\Lambda')^{-1}+\eta)\frac{1}{N(\lambda)}\sharp\left\{j\in S' : A<\lambda_j^2\leq \lambda^2 \text{ and } \alpha_j\geq\epsilon\right\}.$$

It implies that, for λ^2 large enough (depending on η and on A),

$$1-\eta\leq \eta+\epsilon+(L(\Lambda')^{-1}+\eta-\epsilon)\frac{1}{N(\lambda)}\sharp\left\{j\in S' : \lambda_j^2\leq \lambda^2 \text{ and } \alpha_j\geq\epsilon\right\}.$$

In other words, we have shown that, for every $\eta>0$,

$$\lim_{\lambda\rightarrow+\infty}\frac{1}{N(\lambda)}\sharp\left\{j\in S' : \lambda_j^2\leq \lambda^2 \text{ and } \alpha_j\geq\epsilon\right\}\geq \frac{1-2\eta-\epsilon}{L(\Lambda')^{-1}+\eta-\epsilon},$$

which implies the result.

5.3. **Donnay's surfaces.** In this last paragraph, we will deal with the example of Donnay's surfaces that was described in paragraph 3.2. We want to show that proposition 3.2 can be slightly improved in order to allow several subsets, i.e. $|I|\geq 2$ with the notations of paragraph 3.2.

Precisely, we want to verify that α can be chosen >0 in equation (1) for a subsequence of positive density of eigenstates (without giving precise informations on the density of the subset). For that purpose, we pick $\chi\geq 0$ a smooth function which is compactly supported outside the "light-bulb cap" (with χ non identically equal to 0). We also consider a subset $S\subset\mathbb{N}$ of density 1 such that any accumulation point of $(\mu_j)_{j\in S}$ is of the form given by equation (1).

We can apply lemma 2.2 to this function: there exists $S'\subset S$ of density 1 such that any accumulation point of the subsequence $(\mu_j(\chi))_{j\in S'}$ belongs to the interval

$$\left[0,\max_{i\in I}\frac{\int_{\Lambda_i}\chi dL}{L(\Lambda_i)}\right].$$

A trace asymptotics gives us that

$$\lim_{\lambda\rightarrow+\infty}\frac{1}{N(\lambda)}\sum_{j\in S':\lambda_j^2\leq\lambda^2}\mu_j(\chi)=\lim_{\lambda\rightarrow+\infty}\frac{1}{N(\lambda)}\sum_{j:\lambda_j^2\leq\lambda^2}\mu_j(\chi)=\int_{S^*M}\chi dL>0.$$

As $\mu_j(\chi)$ is "almost positive", we can apply the argument of the previous paragraph. In particular, for every $\epsilon>0$ small enough, we obtain the existence of a subset $S_\epsilon\subset S'$ of positive density such that any accumulation point of the sequence $(\mu_j(\chi))_{j\in S_\epsilon}$ belongs to the interval

$$\left[\epsilon,\max_{i\in I}\frac{\int_{\Lambda_i}\chi dL}{L(\Lambda_i)}\right].$$

As $S_\epsilon\subset S$, one knows that any accumulation point of $(\mu_j)_{j\in S_\epsilon}$ is of the form

$$\mu=\alpha\sum_{i\in I}t_i\frac{L_{|\Lambda_i}}{L(\Lambda_i)}+(1-\alpha)\nu_{\text{int}},$$

where $0\leq\alpha\leq 1$, $0\leq t_i\leq 1$, $\sum_{i\in I}t_i=1$ and ν_{int} belongs to the closure of the convex hull of $\{L_\rho:\rho\in\Lambda_{\text{integrable}}\}$. Finally, as $\mu(\chi)\geq\epsilon$ and $\nu_{\text{int}}(\chi)=0$, we find that α must be positive.

Acknowledgements

This work has been partially supported by the Labex CEMPI (ANR-11-LABX-0007-01) and by the grant ANR-09-JCJC-0099-01 of the Agence Nationale de la Recherche. We warmly thank Stephan De Bièvre for discussions and comments related to this work.

References

[1] L. Barreira, Y. Pesin *Smooth ergodic theory and nonuniformly hyperbolic dynamics*, Handbook of Dynamical Systems 1B, 57–263 (2006)

[2] M.V. Berry *Regular and irregular semiclassical wavefunctions*, J. Phys. A: Math. Gen. **10**, 2083 (1977)

[3] N. Burq *Mesures semi-classiques et mesures de défaut*, Séminaire Bourbaki, Astérisque **245**, 167–195 (1997)

[4] Y. Colin de Verdière *Ergodicité et fonctions propres du Laplacien*, Comm. in Math. Phys. **102**, 497–502 (1985)

[5] V. Donnay *Geodesic flow on the two-sphere. I. Positive measure entropy.*, Ergodic Theory Dynam. Systems **8**, 531–553 (1988)

[6] M. Einsiedler, T. Ward *Ergodic Theory: with a view towards Number Theory*, Springer (2011)

[7] J. Galkowski *Quantum Ergodicity for a Class of Mixed Systems*, arXiv:1209.2968 (2012)

[8] B. Gutkin *Note on converse quantum ergodicity*, Proc. AMS **137**, 2795–2800 (2009)

[9] J. Marklof, S. O'Keefe *Weyl's law and quantum ergodicity for maps with divided phase space, with an appendix by S. Zelditch*, Nonlinearity **18**, 277–304 (2005)

[10] J. Marklof, Z. Rudnick *Almost all eigenfunctions of a rational polygon are uniformly distributed*, Journal of Spectral Theory **2**, 107–113 (2012)

[11] I.C. Percival *Regular and irregular spectra*, J. Phys. B: At. Mol. Opt. Phys. **6**, L229–L232 (1973)

[12] Y. Pesin *Characteristic Ljapunov exponents, and smooth ergodic theory*, Russian Math. Surveys **32**, 55–114 (1977)

[13] L. Schwartz *Théorie des distributions*, Hermann (1966)

[14] A. Shnirelman *Ergodic properties of eigenfunctions*, Usp. Math. Nauk. **29**, 181-182 (1974)

[15] A. Shnirelman *Asymptotic properties of eigenfunctions in the regions of chaotic motion*, Addendum to *KAM theory and semiclassical approximations of eigenfunctions* by V.F. Lazutkin, Springer (1993)

[16] J. Sjöstrand *Asymptotic distribution of egenfrequencies for damped wave equations*, Publ. RIMS **36**, 573–611 (2000)

[17] F. Trèves *Topological vector spaces, distributions and kernels.*, Academic Press, New York-London (1967)

[18] S. Zelditch *Uniform distribution of the eigenfunctions on compact hyperbolic surfaces*, Duke Math. Jour. **55**, 919–941 (1987)

[19] S. Zelditch *Quantum ergodicity on the sphere*, Comm. in Mat. Phys. **146**, 61–71 (1992)

[20] S. Zelditch *Recent developments in mathematical quantum chaos*, in Current Developments in Mathematics 2009 (2010)

[21] S. Zelditch *Random orthonormal bases of spaces of high dimension*, arXiv:1210.2069 (2012)

Laboratoire Paul Painlevé (U.M.R. CNRS 8524), U.F.R. de Mathématiques, Université Lille 1, 59655 Villeneuve d'Ascq Cedex, France

E-mail address: `gabriel.riviere@math.univ-lille1.fr`